New Hybrid Two-Step Method for Simulating Lotka-Volterra Model

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ABSTRACT

Simulating Lotka-Volterra model using numerical method require researchers to apply tiny mesh sizes to obtain an accurate result. This approach nevertheless increases the complexity and burden of computer memory and consume long computational time. To overcome these issues, we investigate and construct new two-step solver that could simulate Lotka-Volterra model using bigger mesh size. This paper proposes three new two-step schemes to simulate Lotka-Volterra model. A non-standard approximation scheme with trimean approach was adopted. The nonlinear terms in the model is approximated via trimean approach and differential equation via non-standard denominators. Four sets of parameters were examined to analyse the performance of these new schemes. Results show that these new schemes provide better simulation for large mesh size.

Keywords: Lotka-Volterra, Nonstandard method, Trimean approach, Two-step method

INTRODUCTION

The Lotka-Volterra model was proposed by Alfred James Lotka (1925) and Vito Volterra (1926) in two difference attempt and purposes (Bacaer, 2011). In the model, at least two variables representing the prey and the predators are presented in two differential equations. The prey is assumed to have unlimited source of food, while the only source of food for the predator is the prey. The Lotka-Volterra model has been used in many applications such as transportation (Yuting & Meng, 2011) and security (Yang & Chen, 2015). Jovanović, B., Mostarac, K., Šarac, D., & Rakić, E. (2015) analyse the interaction between corporate and government sectors using Lotka-Volterra, while Yang and Chen (2015) develop a fire security system aimed at awareness of fire security level in university using the same model.
In this paper, three new two-step schemes are developed for simulating the Lotka-Volterra model. The schemes are intended to improve the accuracy of existing numerical approach for big mesh size since numerical method always require the researcher to apply tiny mesh sizes to generate accurate solution. Non-standard approximation and trimean approach are adopted (Mickens, 2003; Yaacob & Hasan, 2015). The trimean (Ji, Wang, Wu, Wu, Xing, & Liang, 2010) concept was used to represent nonlinear terms in Lotka-Volterra model.

MATERIALS AND METHODS

A Lotka-Volterra model with two equations is given as follows.

\[
\frac{dx}{dt} = Ax - Bxy, \quad \frac{dy}{dt} = -Cy + Dxy
\]  

(1)

where \(x\) and \(y\) denote the prey and predator population, \(A > 0\) represent prey birth rate, \(A > 0\) represent rate of prey consumed by predator, \(C > 0\) represent predator death rate and \(D > 0\) denotes predator birth rate.

In this section, three new two step method is constructed. The derivation of all three schemes are discussed and approximation \(\frac{dx}{dt}\) and \(\frac{dy}{dt}\) is done following Mickens (2003), Yaacob and Hasan (2015) and Bhowmik (2009). The derivative is approximated as follows:

\[
\frac{dx}{dt} \approx \frac{x_{i+1} - x_i}{\mathcal{O}}, \quad \frac{dy}{dt} \approx \frac{y_{i+1} - y_i}{\mathcal{O}}
\]

Here, we take the denominator function, \(\mathcal{O} = \sin(h)\). Scheme 1 is developed by replacing \(x\) in eq. (1) at \(i = 0\) with \(x \rightarrow 2x_0 - x_1, xy \rightarrow x_1y_0\), thus

\[
\frac{dx}{dt} = Ax - Bxy \rightarrow \frac{x_1 - x_0}{\mathcal{O}} = A(2x_0 - x_1) - B(x_1y_0)
\]

\[
x_1 - x_0 = A\mathcal{O}(2x_0 - x_1) - B\mathcal{O}(x_1y_0)
\]

\[
x_1 - x_0 = (2A\mathcal{O}x_0 - A\mathcal{O}x_1) - (B\mathcal{O}x_1y_0)
\]

\[
x_1 = \frac{2A\mathcal{O}x_0 + x_0}{1 + A\mathcal{O} + B\mathcal{O}y_0}
\]
The trimean approach is applied at \( i = 1 \), thus

\[
\frac{x_{i+1} - x_i}{\theta} = A \left( \frac{x_{i-1} + 2x_i + x_{i+1}}{4} \right) - B x_{i+1} y_i
\]

\[
x_{i+1} - x_i = 0.25A \theta (x_{i-1} + 2x_i + x_{i+1}) - B \theta (x_{i+1} y_i) \tag{2}
\]

\[
x_{i+1} = \frac{0.25A \theta x_{i-1} + 0.5A \theta x_i + x_i}{(1 - 0.25A \theta + B \theta y_i)}
\]

and by taking \( y \to -y_i + 2y_{i+1}, xy \to 2x_{i+1} y_i - x_i y_{i+1} \), thus

\[
\frac{dy}{dt} = -Cy + Dxy \to \frac{y_{i+1} - y_i}{\theta} = -C(-y_i + 2y_{i+1}) + D(2x_{i+1} y_i - x_i y_{i+1})
\]

at \( i = 0 \)

\[
y_1 = \frac{C \theta y_0 + 2D \theta x_1 y_0 + y_i}{(1 + 2C \theta + D \theta x_0)}.
\]

The trimean approach was applied at \( i = 1 \), thus

\[
\frac{y_{i+1} - y_i}{\theta} = -C \left( \frac{y_{i-1} + 2y_i + y_{i+1}}{4} \right) + D(2x_{i+1} y_i - x_i y_{i+1}) \tag{3}
\]

\[
y_{i+1} = \frac{-0.25C \theta y_{i-1} - 0.5C \theta y_i + 2D \theta x_{i+1} y_i + y_i}{(1 + 0.25C \theta + D \theta x_i)}
\]

Eq. (2) and (3) represent scheme 1.

We develop scheme 2 by replacing \( xy \to 2x_{i+1} y_i - x_i y_{i+2} \), while the other term remains the same as in scheme 1, thus at \( i = 0 \),

\[
\frac{y_1 - y_0}{\theta} = -C(y_i) + D(2x_1 y_{0} - x_0 y_1)
\]

\[
y_1 = \frac{2D \theta x_1 y_0 + y_0}{(1 + C \theta + D \theta x_0)}
\]

The trimean approach was applied at \( i = 1 \) thus

\[
\frac{y_{i+1} - y_i}{\theta} = -C \left( \frac{y_{i-1} + 2y_i + y_{i+1}}{4} \right) + D(2x_{i+1} y_i - x_i y_{i+1}) \tag{4}
\]

\[
y_{i+1} = \frac{-0.25C \theta y_{i-1} - 0.5C \theta y_i + 2D \theta x_{i+1} y_i + y_i}{(1 + 0.25C \theta + D \theta x_i)}
\]
We develop scheme 3 by replacing \(xy \rightarrow 2x_{i+1}y_i - x_{i+1}y_{i+1}\) while other terms remain the same as in scheme 1, thus at \(i = 0\)

\[
\frac{y_1 - y_0}{\varnothing} = -\varnothing(y_1_c) + D(2x_1y_0 - x_1y_1).
\]

\[y_1 = \frac{2D\varnothing x_1y_0 + y_0}{(1 + C\varnothing + D\varnothing x_1)}\]

The trimean approach was applied at \(i = 1\), thus

\[
\frac{y_{i+1} - y_i}{\varnothing} = -\varnothing\left(\frac{y_{i-1} + 2y_i + y_{i+1}}{4}\right) + D(2x_{i+1}y_i - x_{i+1}y_{i+1})
\]

\[y_{i+1} = \frac{-0.25C\varnothing y_{i-1} - 0.5C\varnothing y_i + 2D\varnothing x_{i+1}y_i + y_i}{(1 + 0.25C\varnothing + D\varnothing x_{i+1})}\] (5)

The algorithm for scheme 1 was constructed by using approximate Eq. (2) and (3) and shown in Algorithm 1, while algorithm for scheme 2 was constructed using approximate Eq. (2) and (4) and shown in Algorithm 2, and for scheme 3 using (2) and (5) and shown in Algorithm 3. We conduct an experiment with four sets of parameters (Prian, 2013). The parameters are

a. \(A = 0.4, B = 0.11, C = 0.12, D = 0.0032, h = 0.001, x_0 = 140\), \(y_0 = 6\) \((A > C)\)

b. \(A = 0.4, B = 0.11, C = 0.4, D = 0.0032, h = 0.001, x_0 = 140\), \(y_0 = 6\) \((A = C)\)

c. \(A = 0.4, B = 0.11, C = 0.74, D = 0.0032, h = 0.001, x_0 = 140\), \(y_0 = 6\) \((A < C)\)

d. \(A = 0.4, B = 0.11, C = 0.74, D = 0.0032, h = 1.0, x_0 = 140\), \(y_0 = 6\) \((h = 1.0)\)
New Two-Step method for Lotka-Volterra Model

Algorithm 1: Algorithm for scheme 1
Set  
\[ t_0, x_0, y_0, A, B, C, D, \omega, \tau, \rho, x_0, y_0, h, P \]
Calculate \( x \) and \( y \) using  
\[
x_2 = \frac{2A\omega x_1 + x}{1 + A\omega + B\rho y_1}
\]
\[
y_2 = \frac{C\rho y_1 + 2D\rho x_2 y_1 + y_1}{1 + 2C\rho + D\rho x_1}
\]
Then calculate \( x \) and \( y \) \((i = 2, \ldots, n - 1)\) using  
\[
x_{i+1} = \frac{0.25A\omega x_{i-1} + 0.5A\omega x_i + x_i}{(1 - 0.25A\omega + B\rho y_i)}
\]
\[
y_{i+1} = \frac{-0.25C\rho y_{i-1} - 0.5C\rho y_i + 2D\rho x_{i+1} y_i}{(1 + 0.25C\rho + D\rho x_i)}
\]
Output: \( x_{\min}, x_{\max}, y_{\min}, y_{\max} \) and graphs for prey vs. Predator

Algorithm 2: Algorithm for scheme 2
Set  
\[ t_0, x_0, y_0, A, B, C, D, \omega, \tau, \rho, x_0, y_0, h, P \]
Calculate \( x \) and \( y \) using  
\[
x_2 = \frac{2A\omega x_1 + x}{1 + A\omega + B\rho y_1}
\]
\[
y_2 = \frac{2D\omega x_2 y_1 + y_1}{1 + 2C\rho + D\rho x_1}
\]
Then calculate \( x \) and \( y \) \((i = 2, \ldots, n - 1)\) using  
\[
x_{i+1} = \frac{0.25A\omega x_{i-1} + 0.5A\omega x_i + x_i}{(1 - 0.25A\omega + B\rho y_i)}
\]
\[
y_{i+1} = \frac{-0.25C\rho y_{i-1} - 0.5C\rho y_i + 2D\rho x_{i+1} y_i}{(1 + 0.25C\rho + D\rho x_i)}
\]
Output: \( x_{\min}, x_{\max}, y_{\min}, y_{\max} \) and graphs for prey vs. Predator

Algorithm 3: Algorithm for scheme 3
Set  
\[ t_0, x_0, y_0, A, B, C, D, \omega, \tau, \rho, x_0, y_0, h, P \]
Calculate \( x \) and \( y \) using  
\[
x_2 = \frac{2A\omega x_1 + x}{1 + A\omega + B\rho y_1}
\]
\[
y_2 = \frac{2D\omega x_2 y_1 + y_1}{1 + 2C\rho + D\rho x_1}
\]
Then calculate \( x \) and \( y \) \((i = 2, \ldots, n - 1)\) using  
\[
x_{i+1} = \frac{0.25A\omega x_{i-1} + 0.5A\omega x_i + x_i}{(1 - 0.25A\omega + B\rho y_i)}
\]
\[
y_{i+1} = \frac{-0.25C\rho y_{i-1} - 0.5C\rho y_i + 2D\rho x_{i+1} y_i}{(1 + 0.25C\rho + D\rho x_i)}
\]
Output: \( x_{\min}, x_{\max}, y_{\min}, y_{\max} \) and graphs for prey vs. Predator

Scilab was used to code the algorithms in Algorithm 1-3. We compare our simulated results with that generated using Adam-Bashforth-Moulton method (Hasan, Karim, & Sulaiman, 2015), which is \( O(h^4) \) for accuracy.
RESULTS AND DISCUSSIONS

Graphs plotted by our code are given in Figures 1-4. Figures 1-3 show that all method produces exactly almost similar result. These show the new two-step scheme simulates results which are comparable with that obtained using Adam-Bashforth-Moulton method. However, scheme 1 and 2 produce thicker result compared with others. Even though it produces the same number of fluctuations, the values are quite different.

![Figure 1. Prey vs predator for parameter set 1](image)

Figure 1. Prey vs predator for parameter set 1
Figure 4 shows results simulated for parameter set 4. It is clear schemes 1 and 2 produce the same output, while Adam-Bashforth-Moulton and scheme 3 produce different behaviours. Using larger h does not change the interaction behaviour of prey and predator simulated using Adam-Bashforth-Moulton method. While using larger h does affect the behaviour of our schemes. Schemes 1 and 2 end at its exact equilibrium point, while scheme 3 produces a very interesting behaviour. The exact equilibrium point was calculated assuming that there were no further changes in both predator and prey in time; i.e., by taking $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in Eq. (1) to be zero.

(a) Adam-Bashforth-Moulton

(b) Scheme 1

(c) Scheme 2

(d) Scheme 3

Figure 2. Prey vs predator for parameter set 2
CONCLUSION

This paper showed three new two-step schemes. In order to analyse the performance of these schemes, they are compared with a high accuracy fourth order predictor-corrector method, Adam-Bashforth-Moulton method. The latter method is known for its accuracy; however, it is more complicated than all study's new schemes. For small step size, the present results are comparable

Figure 3. Prey vs predator for parameter set 3
CONCLUSION

This paper showed three new two-step schemes. In order to analyse the performance of these schemes, they are compared with a high accuracy fourth order predictor-corrector method, Adam-Bashforth-Moulton method. The latter method is known for its accuracy; however, it is more complicated than all study’s new schemes. For small step size, the present results are comparable with Adam’s. But for larger step size, scheme 1 and 2 is able to gather the point of equilibrium, while scheme 3 produces extremely interesting behaviour and need further analysis.

In future research, the present authors will apply this algorithm to solve higher order system of ordinary differential equations and other predator-prey models such as Rosenzweig-MacArthur and Beddington-DeAngelis.
ACKNOWLEDGEMENT

The authors acknowledge FRGS/1/2013/ICT07/UKM/02/4 grant for funding this research.

REFERENCES


