Automorphisms of Surface Groups of Genus Two

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ABSTRACT

The mapping class group of a surface is a quotient of the automorphism group $Aut^+\Gamma$ of the surface group by its normal subgroup of inner automorphism. The determination of $Aut^+\Gamma$ may be looked upon as an extension problem. This paper will give a presentation of $Aut^+\Gamma$ where $\Gamma$ is a surface group of genus two. The extension to Seifert fibre group will be given.

INTRODUCTION

In this paper, we consider the surface group

$$\Gamma = <a_1, b_1, \ldots, a_g, b_g | a_1b_1a_1^{-1}b_1^{-1} \ldots a_gb_ga_g^{-1}b_g^{-1} = 1>$$

$\Gamma$ is a fundamental group $\pi_1(T_g)$ of some surface of genus $g$. By Nielsen’s theorem, every automorphism in the fundamental group of a surface is induced by a self-homeomorphism of the surface. With abuse of language, we call those automorphisms induced by the orientation-preserving self-homeomorphisms of the surface, the orientation-preserving automorphisms, denoted by $Aut^+$.

The mapping class group of a surface is the quotient of the automorphism group, $Aut^+\Gamma$, of the surface group, $\Gamma$, by its normal subgroup of inner automorphisms, (Hatcher & Thurston, 1980; Maclachlan & Harvey, 1975). Hence the determination of $Aut^+\Gamma$ may be looked upon as an extension problem.

Although the presentation of the mapping class group of a closed orientable surface of genus $g \geq 2$ have been completed theoretically by Hatcher and Thurston, (1980), the presentation is not laid out explicitly. Hence the determination of $Aut^+\Gamma$ would be very technical. Here we will give an explicit presentation of $Aut^+\Gamma$, where $\Gamma$ is a surface group of genus two.

1. PRELIMINARY

In order to obtain our answer, we need the following presentation of the mapping class group $M(2, 0)$.

Theorem: 1.1

The mapping class group of a surface of genus two, $M(2, 0)$, admits the presentation with generators:
and defining relations:

\[
\omega_i \omega_i + 1 \omega_i = \omega_i + 1 \omega_i \omega_i + 1 , \quad 1 \leq i \leq 5.
\]

Defining relations:

\[
R_i (y_1, y_2, \ldots, y_m) = T_i (x_1, x_2, \ldots, x_n) , \quad 1 \leq i \leq k ,
\]

\[
S_i (x_1, x_2, \ldots, x_n) = 1 , \quad 1 \leq i \leq l ,
\]

\[
y_1y_2^{-1} = T_{ij} (x_1, x_2, \ldots, x_n) , \quad 1 \leq i \leq m , 1 \leq j \leq n ,
\]

for some words \( T_{ij} (x_1, x_2, \ldots, x_n) \).

\[ T_{ij} (x_1, x_2, \ldots, x_n) . \]

### 2. THE RESULT

Let \( \Gamma = \langle a_1, b_1, a_2, b_2 \rangle \) be the automorphisms in \( \Gamma \) induced by \( \omega_i \), that is \( \phi ( \xi_i ) = \omega_i , 1 \leq i \leq 5 \), in the sequence:

\[
1 \rightarrow I ( \Gamma ) \rightarrow \text{Aut}^+ \Gamma \rightarrow M (2,0) \rightarrow 1
\]

Then we have:

\[ \xi_1 : a_1 \rightarrow a_1 b_1 \]

\[ \xi_2 : a_1 \rightarrow a_1 b_1 a_2 \]

\[ \xi_3 : a_1 \rightarrow a_1 b_1 b_2 a_2 \]

\[ \xi_4 : a_1 \rightarrow a_1 b_1 b_2 a_2 \]

where

\[
u (y_i) = w_i , \quad 1 \leq i \leq m
\]
AUTOMORPHISMS OF SURFACE GROUPS OF GENUS TWO

Theorem: 2.1

If $\Gamma$ is a surface group of genus two, then $\text{Aut}^+\Gamma$ is generated by:

$$\xi_i, 1 \leq i \leq 5$$

with defining relations:

$$\xi_1\xi_{i+1}\xi_1 = \xi_{i+1}\xi_1\xi_{i+1}, \quad 1 \leq i \leq 4$$

$$\xi_i\xi_j = \xi_j\xi_i, \quad |i-j| \geq 2$$

$$(\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6)^6 = 1$$

$$(\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_1)^2 = 1$$

Hence, $\text{Aut}^+\Gamma = \text{Aut}^+\Gamma_1$, where

$$\Gamma_1 = \langle \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6 \rangle = 1 : 1 \leq i \leq 6$$

is a Fuchsian group of genus zero with six equal periods, 2.

Proof:

Since $\Gamma$ is centerless, $I(\Gamma) \cong \Gamma$ and hence is finitely presented. We denote the inner automorphisms $I(\xi_i) = \alpha_i, I(\beta_i) = \beta_i$ for $i = 1, 2, 3, 5$. Then by Lemma 1.1, $\text{Aut}^+\Gamma$ is finitely generated by:

$$\alpha_1, \alpha_2, \beta_1, \beta_2, \xi_i \quad 1 \leq i \leq 5$$

Our problem now is reduced to checking the defining relations as in the lemma. We then have the following:

$$\xi_i\xi_{i+1}\xi_1 = \xi_{i+1}\xi_1\xi_{i+1}, \quad 1 \leq i \leq 4$$

$$\xi_i\xi_j = \xi_j\xi_i, \quad |i-j| \geq 2$$

$$(\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6)^6 = 1$$

$$(\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_1)^2 = 1$$

$$\alpha_1 \xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_1 = 1$$

$$\alpha_1 \xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_1 = 1$$

where $[a, b] = aba^{-1}b^{-1}$

Now let $\gamma = \xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_1$, then we have:

$$\alpha_1 = \xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_1$$

$$\beta_1 = \xi_2\alpha_1\xi_2\alpha_1$$

$$\beta_2 = \alpha_1\xi_2\beta_1\xi_3\beta_1^{-1}$$

$$\alpha_2 = \beta_2\xi_4\beta_2\xi_4$$

Hence $\text{Aut}^+\Gamma$ is generated only by $\xi_i, 1 \leq i \leq 5$.

The relations (2.2) can be shown to be redundant and thus can be eliminated giving the result.

The last part of the theorem follows from the presentation of $\text{Aut}^+\Gamma_1$. (See Abu Osman.)
Maclachlan, (1974), proved the following:

**Theorem: 2.2**

If $\Gamma$ has a signature $(g; -; s)$ and $\Gamma^0$ is a uniformizing group of $\frac{\text{IH}/\Gamma}{x_0}$, then $\text{Aut}^+\Gamma$ is isomorphic to a subgroup of index $(s+1)$ in $\text{Aut}^+\Gamma/\Gamma(\Gamma^0)$, where $\text{IH}$ is the upper half of the complex plane.

In our case, $\Gamma$ is such that $g = 2$ and $s = 0$, and hence $\text{Aut}^+\Gamma = \frac{\text{Aut}^+\Gamma^0}{\Gamma(\Gamma^0)} = \text{M}(2, 1)$ the mapping class group of a surface of genus two with one puncture. Hence we have shown that:

**Theorem: 2.3**

If $\Gamma$ is a surface group of genus two, then

$$\text{Aut}^+\Gamma \cong \text{M}(2, 1)$$

**Remarks: 2.1**

By the above theorem, we now know the presentation of $\text{M}(2, 1)$.

If $\Gamma^* = \langle a_1, b_1, a_2, b_2 | (a_1 b_1 a_1 b_1 a_2 b_2 a_2 b_2)^n = 1 \rangle$, then $\text{M}(2, 1) \cong \text{Aut}^+\Gamma^*/\Gamma(\Gamma)^\phi$. Using the same procedure, we can get the presentation of $\text{Aut}^+\Gamma^*$. (Abu Osman, 1984).

### 3. THE EXTENSION TO SEIFERT FIBRE GROUP

Let $\Gamma$ be a Fuchsian group of genus two as in 2. Let $G$ be a central extension group of $\mathbb{Z}$, by $\Gamma$ such that:

$$G = \langle a_1, b_1, a_2, b_2, z | a_1 b_1 a_1 b_1 a_2 b_2 a_2 b_2 z^n, z \rightarrow a_1 b_1 \rangle$$

(3.1)

where $\leftrightarrow$ denoted commutativity.

G is called the Siefert fibre group which is the fundamental $\pi_2(M)$ of the Seifert manifold (Orlik, 1972)

We denote $\text{Aut}^+G$ those automorphisms in $G$ induced by the automorphisms in $\text{Aut}^+\Gamma$ and $\Psi$ induced $\Psi_\phi : \text{Aut}^+G \rightarrow \text{Aut}^+\Gamma$ in a natural way. Let $N = \text{ker} \, \Psi_\phi$. We then have:

$$1 \rightarrow N \rightarrow \text{Aut}^+G \rightarrow \Psi_\phi \rightarrow \text{Aut}^+\Gamma \rightarrow 1.$$  

It is easy to see that $N$ is guaranted by $\tau_j$, $1 \leq j \leq 4$, defined as:

$$\tau_1 : a_1 \rightarrow za_1 a_1 \rightarrow a_1$$

$$a_2 \rightarrow a_2$$

$$b_1 \rightarrow b_1$$

$$b_2 \rightarrow b_2$$

$$\tau_2 : a_1 \rightarrow a_1 \rightarrow za_2 a_2$$

$$b_1 \rightarrow b_1$$

$$b_2 \rightarrow b_2$$

$$\tau_3 : a_1 \rightarrow b_1$$

$$a_2 \rightarrow a_2$$

$$b_1 \rightarrow zb_1$$

$$b_2 \rightarrow b_2$$

$$\tau_4 : a_1 \rightarrow a_1$$

$$a_2 \rightarrow a_2$$

$$b_1 \rightarrow b_1$$

$$b_2 \rightarrow zb_2.$$  

Observe that for every $i$ and $j$, $\tau_i \tau_j = \tau_j \tau_i$. Hence $N$ is a free abelian group $\mathbb{Z}^4$. By Lemma, we can find the presentation of $\text{Aut}^+G$; since we know the presentation $\text{Aut}^+\Gamma$.

For each $i$, $1 \leq i \leq 5$, pick $\delta_i : \text{Aut}^+G$, the lift $\xi_i \circ \text{Aut}^+\Gamma$, in such a way that $\delta_i$ map $a_j, b_j$ in $\mathbb{Z}$, in exactly the same manner as $\xi_i$ and map $z$ into $z$. Then $\Psi_\phi (\delta_i) = \xi_i$, $1 \leq i \leq 5$. Hence,

$$\delta_i, \quad 1 \leq i \leq 5$$

(3.2)

$$\tau_j, \quad 1 \leq j \leq 4$$
generate \( \text{Aut}^+ G \). The defining relations induced by the defining relations of \( \text{Aut}^+ T \) are exactly the same due to our choice of \( \delta \). The relations arising from \( N \) are \( \tau_i \tau_j = \tau_j \tau_i \) for every \( i \) and \( j \). The other relations can easily be calculated. Hence we have the following:

**Theorem: 3.1**

If \( G \) is a Seifert fibre group with presentation (3-1), then \( \text{Aut}^+ G \) is generated by:

\[
\begin{align*}
\delta_i, & \quad 1 \leq i \leq 5 \\
\tau_j, & \quad 1 \leq j \leq 4
\end{align*}
\]

with defining relations:

\[
\begin{align*}
\delta_i \delta_{i+1} \delta_i &= \delta_{i+1} \delta_1 \delta_{i+1}, & 1 \leq i \leq 4 \\
\delta_i \delta_j &= \delta_j \delta_i, & |i - j| \geq 2 \\
(\delta_1 \delta_2 \delta_3 \delta_4 \delta_5)^6 &= 1 \\
(\delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_4 \delta_3 \delta_2 \delta_1)^2 &= 1 \\
\tau_i &\longrightarrow \tau_j, \text{ for all } i \text{ and } j.
\end{align*}
\]

\[
\begin{align*}
\delta_i \longrightarrow \tau_1, \tau_3, \tau_4 \\
[\delta_1, \tau_2] &= \tau_3 \\
\delta_2 \longrightarrow \tau_1, \tau_2, \tau_4 \\
[\delta_2^{-1}, \tau_3] &= \tau \\
\delta_3 \longrightarrow \tau_2, \tau_3, \tau_4 \\
[\delta_3, \tau_1] &= \tau_3 \tau_2 \\
\delta_4 \longrightarrow \tau_1, \tau_3, \tau_4 \\
[\delta_4, \tau_2] &= \tau_4 \\
\delta_5 \longrightarrow \tau_1, \tau_2, \tau_3 \\
[\tau_4, \delta_5] &= \tau_2
\end{align*}
\]

where \( \tau \) denoted commutativity and \( [a, b] = aba^{-1}b^{-1} \).

**REFERENCES**


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